

ON DISPERSIVE BLOW-UPS FOR THE NONLINEAR SCHRÖDINGER EQUATION

YOUNGHUN HONG AND MAJA TASKOVIC

ABSTRACT. In this article, we provide a simple method for constructing dispersive blow-up solutions to the nonlinear Schrödinger equation. Our construction mainly follows the approach in Bona, Ponce, Saut and Sparber [2]. However, we make use of the dispersive estimate to enjoy the smoothing effect of the Schrödinger propagator in the integral term appearing in Duhamel's formula. In this way, not only do we simplify the argument, but we also reduce the regularity requirement to construct dispersive blow-ups. In addition, we provide more examples of dispersive blow-ups by constructing solutions that blow up on a straight line and on a sphere.

1. INTRODUCTION

In this paper, we extend the recent work of Bona, Ponce, Saut and Sparber [2] by lowering the regularity requirement and constructing few more examples of dispersive blow-up solutions for the nonlinear Schrödinger equation (NLS) in \mathbb{R}^d with $d \geq 2$:

$$i\partial_t u + \Delta u \pm |u|^{p-1}u = 0, \quad u(0) = u_0, \quad (1.1)$$

where $u = u(t, x) : I \times \mathbb{R}^d \rightarrow \mathbb{C}$ and $I \subset \mathbb{R}$.

The term dispersive blow-up, abbreviated by DBU, was coined by Bona and Saut [3] to describe smooth solutions that develop a blow up at a single point. To be precise, we say that $u(t, x)$ has a *dispersive blow-up* at a point $(t_*, x_*) \in \mathbb{R} \times \mathbb{R}^d$ if

$$\lim_{(t,x) \rightarrow (t_*, x_*)} |u(t, x)| = +\infty, \quad (1.2)$$

and if $u(t, x)$ is continuous everywhere except at (t_*, x_*) . In other words, a dispersive blow up is a point-wise phenomena, while usual blow-up solutions to the dispersive PDEs typically blow-up in a certain Sobolev norm.

The theory of the dispersive singularity has its roots in the paper by Benjamin, Bona and Mahony [1], where it is remarked that the linearized Korteweg-de Vries equation (KdV) can develop singularity at a single point. Bona and Saut [3] further studied dispersive blow-ups in depth in the context of KdV. They proved the existence of solutions of KdV that develop point singularities in finite time due to the focusing effect related to the dispersive properties of the equation.

In recent years, dispersive blow-ups were extended to the framework of the Schrödinger equation. In [4], Bona and Saut analyzed dispersive blow-ups for the one-dimensional linear and nonlinear Schrödinger equations. Later, Bona, Ponce, Saut and Sparber [2] extended the theory to multiple dimensions and relaxed restriction on the nonlinearity.

Date: January 25, 2016.

AMS Subject Classification: 35Q55 (35B44, 35L67, 35L70).

Mailing address: The University of Texas at Austin, Department of Mathematics, RLM 8.100, 2515 Speedway Stop C1200, Austin, Texas 78712.

The general strategy for constructing dispersive blow-ups in [1, 3, 4, 2] consists of two steps. The first step is to construct a dispersive blow-up to the linear equation by choosing an appropriate smooth initial data explicitly. The next step is prove that the same initial data leads to dispersive blow-up for the nonlinear problem by showing that the nonlinearity does not destroy the formation of the singularity. The formation of a singularity is associated only with the dispersion relation in the linear part of the equation. This is the reason that such a singularity formation is called a “dispersive” blow-up.

In this paper, following the general strategy in [1, 3, 4, 2], we construct dispersive blow-ups for NLS at an arbitrary point. Throughout the paper, we make the following assumption, which is natural since it guarantees local well-posedness of the equation in $H^s(\mathbb{R}^d)$, see Propostion 2.2.

Assumption 1.1. *Assume*

$$s > \frac{d}{2} - \frac{2}{p-1}, \quad s \geq 0, \quad \text{and } p > 1. \quad (1.3)$$

When p is not an even integer or when $s \geq 2$, we further assume that

$$\begin{cases} p > s - 1, & \text{if } 2 \leq s < 4, \text{ and } 1 < s < \frac{d}{2} \\ p > s - 2, & \text{if } s \geq 4, \text{ and } 1 < s < \frac{d}{2} \\ p > 1 + \lfloor s \rfloor, & \text{if } s \geq \frac{d}{2}. \end{cases} \quad (1.4)$$

Theorem 1.2 (DBU at a point). *Let $(t_*, x_*) \in \mathbb{R} \times \mathbb{R}^d$ with $d \geq 2$. For any $s \in (\frac{d}{2} - \frac{2}{p}, \frac{d}{2}]$, for which p and s satisfy the conditions in the Assumption 1.1, there exists initial data $u_0 \in H^s(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$ such that the solution $u(t)$ to NLS with initial data u_0 blows up at (t_*, x_*) , that is,*

$$\lim_{(t,x) \rightarrow (t_*, x_*)} |u(t, x)| = +\infty.$$

Moreover, $u(t, x)$ is continuous in $\mathbb{R} \setminus \{t_*\} \times \mathbb{R}^d$, and $u(t_*, \cdot)$ is continuous on $\mathbb{R}^d \setminus \{x_*\}$.

Remark 1.3. The assumption $s \leq \frac{d}{2}$ is necessary since dispersive blow up does not exist in H^s when $s > \frac{d}{2}$ due to the embedding $H^s \hookrightarrow L^\infty$ for $s > \frac{d}{2}$.

The novelty of this theorem lies in its simpler proof. In particular, we use dispersive estimate to show that the nonlinear part does not affect the blow up of the linear part. As a consequence we lower regularity requirement on the initial data.

We also provide new examples of dispersive blow-ups by constructing solutions that blow up along a stright line in \mathbb{R}^d .

Theorem 1.4 (DBU along a line). *Let $t_* \in \mathbb{R}$ and let l be a straight line in \mathbb{R}^d . Suppose that $1 < p < 4$. For any $s \in (\frac{d}{2} - \frac{2}{p}, \frac{d}{2}]$, for which p and s satisfy the conditions in the Assumption 1.1, there exists initial data $u_0 \in H^s(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$ such that the corresponding solution u to NLS blows up along the line l at time $t = t_*$, that is, for any $x_* \in l$*

$$\lim_{(t,x) \rightarrow (t_*, x_*)} |u(t, x)| = +\infty.$$

Moreover, $u(t, x)$ is continuous on $\mathbb{R} \setminus \{t_*\} \times \mathbb{R}^d$, and $u(t_*, \cdot)$ is continuous on $\mathbb{R}^d \setminus l$.

It is of interest to construct a dispersive blow-up on a compact manifold. As an example, we construct a DBU on the unit sphere in \mathbb{R}^3 .

Theorem 1.5 (DBU on a sphere in \mathbb{R}^3). *Let $t_* \in \mathbb{R}$. Suppose also that $1 < p < 2$. Then for any positive $s \in (\frac{3}{2} - \frac{2}{p}, \frac{3}{2}]$, there exists initial data $u_0 \in H^s(\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3)$ such that the solution u to NLS with initial data u_0 blows up along the sphere \mathcal{S}^2 at time $t = t_*$, that is, for any $x_* \in \mathcal{S}^2$*

$$\lim_{(t,x) \rightarrow (t_*, x_*)} |u(t, x)| = +\infty.$$

Moreover, $u(t, x)$ is continuous on $\mathbb{R} \setminus \{t_*\} \times \mathbb{R}^3$, and $u(t_*, \cdot)$ is continuous on $\mathbb{R}^3 \setminus \mathcal{S}^2$.

The key ingredient for proving Theorem 1.2, 1.4 and 1.5 is the smoothing estimate in Proposition 2.4. It will be used in the second step of the general strategy, that is, for controlling the nonlinear term. Proposition 2.4 simplifies and improves the corresponding proposition in Bona-Ponce-Saut-Sparber [2] (Proposition 4.1) in lowering the regularity requirement from $s \in (\frac{d}{2} - \frac{1}{2(p-1)}, \frac{d}{2}]$ to $s \in (\frac{d}{2} - \frac{2}{p}, \frac{d}{2}]$. This is achieved via the use of the dispersive estimate.

We remark that reducing regularity requirement in the smoothing estimate is not just a purely mathematical issue. It is helpful for including higher power nonlinearities in construction of a dispersive blow-up along a line. Indeed, by the argument in Section 3 with the previously known smoothing estimate in [2], one can construct a nonlinear dispersive blow-up along a line only for sub-quadratic nonlinearity, i.e., $1 < p < 2$. It is due to the fact that in the construction, we use an initial data making a dispersive blow-up in \mathbb{R}^{d-1} (see (3.2)). However, by the smoothing estimate in Proposition 2.4, one can include any nonlinearity $1 < p < 4$. Similarly, the condition on p in Theorem 1.5 is also relaxed by a smoothing estimate with a low regularity.

Outline of the paper. We prove Theorem 1.2 in Section 2, Theorem 1.4 in Section 3. and Theorem 1.5 in Section 4.

2. CONSTRUCTION OF A DBU AT A POINT: PROOF OF THEOREM 1.2

In this section, we prove the main theorem (Theorem 1.2) following the approach in Bona, Ponce, Saut and Sparber [2]. Before we begin, we note that both linear and nonlinear Schrödinger equations are invariant under the spatial translation, i.e., if $u(t, x)$ solves the equation, then $u(t, x - x_*)$ solves the same equation for any $x_* \in \mathbb{R}^d$. Thus, it suffices to construct a DBU that blows up at the origin in \mathbb{R}^d .

The proof of Theorem 1.2 proceeds in two steps: (1) Construction of a DBU for the linear equation. (2) Proving that the nonlinearity does not break singularity formation.

2.1. Construction of a linear DBU. Consider the linear Schrödinger equation

$$i\partial_t u + \Delta u = 0, \quad u(0) = u_0. \quad (2.1)$$

By the following proposition, one can explicitly construct a DBU to the linear equation (2.1) at the origin at an arbitrary time.

Proposition 2.1 (Linear DBU at a point; [2], Lemma 2.1). *Let $u(t)$ be the solution to the linear Schrödinger equation with initial data*

$$u_0(x) = \frac{\alpha e^{-i|x|^2/4t_*}}{(1 + |x|^2)^m} \in C^\infty(\mathbb{R}^d) \cap L^2(\mathbb{R}^d), \quad (2.2)$$

where $\alpha \in \mathbb{R}$, $t_* \in \mathbb{R}$ and $m \in (\frac{d}{4}, \frac{d}{2}]$. Then, $u(t, x)$ blows up at $(t_*, 0)$, i.e., $|u(t, x)| \rightarrow +\infty$ as $(t, x) \rightarrow (t_*, 0)$. Moreover, $u(t, x)$ is continuous in $\mathbb{R} \setminus \{t_*\} \times \mathbb{R}^d$, and $u(t_*, x)$ is continuous in $\mathbb{R}^d \setminus \{0\}$.

Sketch of the Proof. The well-known representation of the solution to the linear problem (2.1)

$$u(t, x) = \frac{1}{(4\pi it)^{d/2}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{4t}} u_0(y) dy, \quad (2.3)$$

implies that

$$u(t_*, x) = \frac{1}{(4\pi it_*)^{d/2}} e^{\frac{i|x|^2}{4t_*}} \int_{\mathbb{R}^d} e^{\frac{-2ix \cdot y}{4t_*}} \frac{\alpha}{(1 + |y|^2)^m} dy.$$

The above integral is the Fourier transform of a Bessel potential (see e.g. [10]). Therefore

$$u(t_*, x) = \frac{C\alpha}{(4\pi it_*)^{d/2}} e^{\frac{i|x|^2}{4t_*}} \left| \frac{x}{4t_*} \right|^{-\nu} K_\nu\left(\left| \frac{x}{4t_*} \right|\right), \quad (2.4)$$

where $K_\nu(|x|)$ is the Fourier transform of $\frac{1}{(1+|y|^2)^m}$ and $\nu = \frac{d}{2} - m$. By the properties of the Bessel potential, K_ν behaves as $K_\nu(|x|) \sim \frac{C}{|x|^\nu}$ for $\nu > 0$ and $K_0(|x|) \sim -\log(|x|)$ as $x \rightarrow 0$. Therefore

$$\begin{aligned} u(t_*, x) &\sim |x|^{-(d-2m)}, \quad \text{as } x \rightarrow 0, \quad \text{if } d - 2m > 0, \\ u(t_*, x) &\sim -\log(|x|) \quad \text{as } x \rightarrow 0, \quad \text{if } d - 2m = 0. \end{aligned}$$

Since $2m \leq d$, we conclude that $u(t_*, 0) = \infty$. Also, by (2.4) and the properties of the Bessel potential, the continuity of $u(t_*, x)$ in $\mathbb{R}^d \setminus \{0\}$ follows. The continuity of $u(t, x)$ in $\mathbb{R} \setminus \{t_*\} \times \mathbb{R}^d$ can be proved as in [[4], Theorem 2.1.]. \square

2.2. Construction of a nonlinear DBU. Now, we consider the Cauchy problem for NLS (1.1). It is well-known that this equation is locally well-posed in a sub-critical space (see [5] for instance, and [9] for relaxed conditions in case of non-integer p). To be precise, we define the solution to the NLS with initial data u_0 as the solution to the integral equation

$$u(t) = e^{it\Delta} u_0 \pm i \int_0^t e^{i(t-s)\Delta} (|u|^{p-1} u)(s) ds. \quad (2.5)$$

We call (q, r) *admissible* if $2 \leq q, r \leq \infty$, $(q, r, d) \neq (2, \infty, 2)$ and

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}.$$

Proposition 2.2 (Local well-posedness in $H^s(\mathbb{R}^d)$, [5, 9]). *Let p and s satisfy the conditions in Assumption 1.1. For such choice of s , we have:*

- (i) *If $u_0 \in H^s$, then there exist a time interval $I = I(u_0)$ and a unique solution $u(t) \in C_{t \in I} H_x^s \cap L_{t \in I}^q W_x^{s, r}$ to the NLS with initial data u_0 for all admissible pairs (q, r) .*
- (ii) *If $\|u_0\|_{H^s}$ is sufficiently small, then $u(t)$ exists in H^s globally in time.*

Following the approach in Bona-Ponce-Saut-Sparber [2], we aim to show that the initial data u_0 , given by (2.2), produces a DBU for the nonlinear problem at the same point and at the same time as for the linear problem. To this end, first, we need to make sure that $u_0 \in H^s$ for $s > \frac{d}{2} - \frac{2}{p-1}$, because otherwise, we do not know that the solution to NLS exists. The following lemma provides a sufficient condition.

Lemma 2.3 ([2], Lemma 3.3.). *The initial data u_0 , given in (2.2), is contained in H^s if $2m > s + \frac{d}{2}$.*

Next, we show that the integral term in the Duhamel formula (2.5) stays bounded during the existence time.

Proposition 2.4 (Smoothing estimate). *Let $s > \frac{d}{2} - \frac{2}{p}$. Suppose that $u(t)$ solves NLS (1.1) on a time interval I . Then, we have*

$$\left\| \int_0^t e^{i(t-s)\Delta} (|u|^{p-1}u)(s) ds \right\|_{L_{t \in I}^\infty C_x^0} \leq |I|^{\frac{4+p(2s-d)}{6}} \|u\|_{L_{t \in I}^\infty H_x^s}^p. \quad (2.6)$$

The proof of the smoothing estimate is based on the dispersive estimate.

Lemma 2.5 (Dispersive estimate). *For $2 \leq r \leq \infty$, we have*

$$\|e^{it\Delta} f\|_{L^r(\mathbb{R}^d)} \lesssim |t|^{-d(\frac{1}{2} - \frac{1}{r})} \|f\|_{L^{r'}(\mathbb{R}^d)},$$

where r' is a Hölder conjugate of r , that is $\frac{1}{r} + \frac{1}{r'} = 1$.

Proof of Proposition 2.4. Let $\epsilon > 0$ be an arbitrarily small number such that $\frac{dp-4+3\epsilon}{2p} \leq s$. Then, applying the Sobolev inequality, Minkowski inequality and the dispersive estimate, respectively, we obtain

$$\begin{aligned} \left\| \int_0^t e^{i(t-s)\Delta} F(s) ds \right\|_{L_{t \in I}^\infty C_x^0} &\lesssim \left\| \int_0^t e^{i(t-s)\Delta} F(s) ds \right\|_{L_{t \in I}^\infty W_x^{\frac{d-2}{2} + \epsilon, \frac{2d}{d-2+\epsilon}}} \\ &\leq \left\| \int_0^t \|e^{i(t-s)\Delta} F(s)\|_{W_x^{\frac{d-2}{2} + \epsilon, \frac{2d}{d-2+\epsilon}}} ds \right\|_{L_{t \in I}^\infty} \\ &\lesssim \left\| \int_0^t \frac{1}{|t-s|^{\frac{2-\epsilon}{2}}} \|F(s)\|_{W_x^{\frac{d-2}{2} + \epsilon, \frac{2d}{d+2-\epsilon}}} ds \right\|_{L_{t \in I}^\infty}. \end{aligned}$$

Note that the parameters in the above Sobolev inequality were chosen so that the power of $|t-s|$ obtained after application of the dispersive estimates is just barely integrable. We next integrate the power of $|t-s|$, and obtain

$$\begin{aligned} \left\| \int_0^t e^{i(t-s)\Delta} F(s) ds \right\|_{L_{t \in I}^\infty C_x^0} &\lesssim \left(\int_0^t \frac{1}{|t-s|^{\frac{2-\epsilon}{2}}} ds \right) \|F\|_{L_{t \in I}^\infty W_x^{\frac{d-2}{2} + \epsilon, \frac{2d}{d+2-\epsilon}}} \\ &\lesssim |I|^{\frac{\epsilon}{2}} \|F\|_{L_{t \in I}^\infty W_x^{\frac{d-2}{2} + \epsilon, \frac{2d}{d+2-\epsilon}}}. \end{aligned} \quad (2.7)$$

Now, by applying the fractional Leibnitz rule to (2.7) we obtain

$$\left\| \int_0^t e^{i(t-s)\Delta} (|u|^{p-1}u)(s) ds \right\|_{L_{t \in I}^\infty C_x^0} \lesssim |I|^{\frac{\epsilon}{2}} \|u\|_{L_{t \in I}^\infty W_x^{\frac{d-2}{2} + \epsilon, \frac{2dp}{p(d-2+2\epsilon)-3\epsilon+4}}}^{p-1} \|u\|_{L_{t \in I}^\infty L_x^{\frac{2dp}{(p-1)(4-3\epsilon)}}}.$$

Finally, another application of the Sobolev inequality leads to

$$\left\| \int_0^t e^{i(t-s)\Delta} (|u|^{p-1}u)(s) ds \right\|_{L_{t \in I}^\infty C_x^0} \lesssim |I|^{\frac{\epsilon}{2}} \|u\|_{L_{t \in I}^\infty H_x^{\frac{dp-4+3\epsilon}{2p}}}^p.$$

□

Remark 2.6. (i) The proof of Proposition 2.4 is very simple, but it improves the corresponding proposition in Bona, Ponce, Saut and Sparber [2] (see Proposition 4.1) in lowering the regularity requirement. The key in the proof is a Kato smoothing type estimate (2.7). This estimate is related to the fact that the inhomogeneous term

$$\int_0^t e^{i(t-s)\Delta} F(s) ds$$

satisfies better Strichartz estimates (see [6], for example). In (2.7), we exploit the smoothing property of the linear Schrödinger flow by the dispersive estimate as long as the bound is integrable in time.

(ii) The smoothing estimate (2.6) is sharp (except the endpoint case $s = \frac{d}{2} - \frac{2}{p}$) in the sense that if $s < \frac{d}{2} - \frac{2}{p}$, the inequality (2.6) is super-critical.

Proof of Theorem 1.2. Fix any $t_* \in \mathbb{R}$. Combining conditions on s, p, d and m in Proposition 2.1, 2.2 and 2.4 and Lemma 2.3, we have

$$\max \left\{ \frac{d}{2} - \frac{1}{p}, \frac{d}{4} \right\} < m \leq \frac{d}{2}.$$

It is obvious that one can always find m that satisfies these bounds. Pick any such m and let u_0 be initial data defined by (2.2). Let $u(t)$ be the solution to NLS with initial data u_0 . Here, by choosing sufficiently small α (see (2.2)), one may assume that t_* is contained in the existence time of $u(t)$. Then, as a consequence of Proposition 2.1 and 2.4, the linear part in the integral equation (2.5) blows up at $(t_*, 0)$ and the nonlinear part remains bounded during the existence time. Therefore, we conclude that $u(t, x)$ blows up at $(t_*, 0)$. \square

3. CONSTRUCTION OF A DBU ALONG A LINE: PROOF OF THEOREM 1.4

In this section, we provide a new example of a dispersive blow-up, namely along a line. Since both linear and nonlinear Schrödinger equations are invariant under the spatial translation and rotation, it suffices to construct a DBU along the line

$$l_1 := \{(x_1, 0, \dots, 0) \in \mathbb{R}^d : x_1 \in \mathbb{R}\}. \quad (3.1)$$

Following the general strategy, the DBU is first constructed for the linear problem. The idea is to find initial data that leads to a solution of the linear Schrödinger equation which has a dispersive blow up in $d - 1$ variables at $(x_2, x_3, \dots, x_d) = (0, 0, \dots, 0)$ and which is regular in the first variable x_1 . This will yield a DBU for the linear problem along the straight line $\{(x_1, 0, \dots, 0) \in \mathbb{R}^d : x_1 \in \mathbb{R}\}$. Finally, the smoothing estimate Proposition 2.4 is used to control the nonlinear part in the NLS and show that the same initial data generates a DBU solution for the nonlinear problem as well.

Throughout this section, we use the notation

$$x = (x_1, x_2, \dots, x_d) = (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{d-1}, \quad \bar{x} = (x_2, \dots, x_d) \in \mathbb{R}^{d-1}.$$

3.1. Construction of a linear DBU. As a first step, we construct initial data that leads to a DBU solution to the linear problem (2.1) that blows up along the line l_1 (3.1). The idea is to use initial data $u_0(x) = u_0(x_1, \bar{x})$ that separates variables x_1 and \bar{x} in such a way that the factor depending on x_1 is regular, while the factor depending on \bar{x} develops DBU at the point $\bar{0} \in \mathbb{R}^{d-1}$. Such an initial data will lead to DBU along the line l_1 .

Proposition 3.1 (Linear DBU along a straight line). *For any $y_1 \in \mathbb{R}$, any $\bar{y} \in \mathbb{R}^{d-1}$ and $m \in (\frac{d-1}{4}, \frac{d-1}{2}]$, define the regular and the blow-up factor respectively by*

$$\varphi_{\text{reg}}(z) = e^{-z^2}, \quad \varphi_{\text{DBU}}(\bar{y}) = \left(\frac{1}{(1 + |\bar{y}|^2)^m} \right)^\wedge(\bar{y}).$$

For some $\alpha \in \mathbb{R}$ and any $y \in \mathbb{R}^d$ let

$$\varphi_0(y) = \alpha \varphi_{\text{reg}}(y_1) \varphi_{\text{DBU}}(\bar{y}). \quad (3.2)$$

Define the initial data u_0 by

$$u_0(x) = e^{-\frac{i|x|^2}{4t_*}} \varphi_0^\vee(x) = \left(e^{-z^2} \right)^\vee(x_1) \frac{\alpha e^{-\frac{i|x|^2}{4t_*}}}{(1 + |\bar{x}|^2)^m}. \quad (3.3)$$

Let $u(t)$ be the solution to the linear Schrödinger equation with the initial data u_0 (3.3). Then, $u(t, x)$ blows up along the line l_1 (3.1), that is, for any $x_* \in l_1$, $|u(t, x)| \rightarrow +\infty$ as $(t, x) \rightarrow (t_*, x_*)$. Moreover, $u(t, x)$ is continuous in $\mathbb{R} \setminus \{t_*\} \times \mathbb{R}^d$, and $u(t_*, x)$ is continuous in $\mathbb{R}^d \setminus l_1$.

Proof. The function φ_{reg} is smooth, while the function $\varphi_{\text{DBU}}(\bar{y})$ is a Fourier transform of a Bessel potential. To ensure that $\varphi_{\text{DBU}}(\bar{y})$ has singularity only at $\bar{0}$, we need $m \leq \frac{d-1}{2}$ similarly to the discussion in Section 2. We also need that $m > \frac{d-1}{4}$ so that $\varphi_{\text{DBU}}^\vee$ is square integrable. Thus the bounds on m in the statement of the proposition.

Using the representation (2.3) for the solution $u(t, x)$, we get

$$\begin{aligned} u(t_*, x) &= \frac{1}{(4\pi i t_*)^{\frac{d}{2}}} e^{\frac{i|x|^2}{4t_*}} \int_{\mathbb{R}^d} e^{-iy \cdot \frac{x}{2t_*}} \varphi_0^\vee(y) dy \\ &= \frac{e^{\frac{i|x|^2}{4t_*}}}{(4\pi i t_*)^{\frac{d}{2}}} \varphi_0\left(\frac{x}{2t_*}\right) \\ &= \frac{\alpha e^{\frac{i|x|^2}{4t_*}}}{(4\pi i t_*)^{\frac{d}{2}}} \varphi_{\text{reg}}\left(\frac{x_1}{2t_*}\right) \varphi_{\text{DBU}}\left(\frac{\bar{x}}{2t_*}\right). \end{aligned}$$

Since $\varphi_{\text{reg}}(\frac{x_1}{2t_*})$ is bounded for all $x_1 \in \mathbb{R}$ and $\varphi_{\text{DBU}}(\bar{x})$ blows up as $\bar{x} \rightarrow \bar{0}$, we conclude that for any $x_1 \in \mathbb{R}$,

$$|u(t_*, x_1, \bar{x})| \rightarrow +\infty \quad \text{as } \bar{x} \rightarrow \bar{0}.$$

Thus, initial data u_0 leads to a solution with dispersive blow up along the straight line l_1 . The continuity of $u(t_*, x)$ in $\mathbb{R}^d \setminus \{l_1\}$ follows by properties of Bessel potential φ_{DBU} and smoothness of φ_{reg} . The continuity of $u(t, x)$ in $\mathbb{R} \setminus \{t_*\} \times \mathbb{R}^d$ can be proved as in [[4], Theorem 2.1.]. \square

3.2. Construction of a nonlinear DBU. We now show that the the initial data u_0 (3.3) will lead to the dispersive blow ups solution for the NLS as well, along the same line and at the same time.

Proof of Theorem 1.4. Fix any $t_* \in \mathbb{R}$. Due to the dimension reduction in the DBU part of the initial data (3.3), Lemma 2.3 implies that $u_0 \in H^s(\mathbb{R}^d)$ if $2m > s + \frac{d-1}{2}$. Combining this

with $m \in (\frac{d-1}{4}, \frac{d-1}{2}]$ with conditions on s, p, d and m in Proposition 2.2 and Proposition 2.4 we get

$$\max \left\{ \frac{d}{2} - \frac{1}{p} - \frac{1}{4}, \frac{d-1}{4} \right\} < m \leq \frac{d-1}{2}.$$

m that satisfies these bounds always exists. Pick one such m and let u_0 be the initial data given by the formula (3.3). By choosing sufficiently small α (see (3.2)), one may assume that t_* is contained in the interval of existence for the solution to NLS with initial data u_0 . Then, as a consequence of Proposition 3.1 and 2.4, the linear part in the integral equation (2.5) blows up at $(t_*, 0)$ and the nonlinear part remains bounded during the existence time. Therefore, we conclude that $u(t, x)$ blows up at $(t_*, 0)$. \square

4. CONSTRUCTION OF A DBU ON A SPHERE IN \mathbb{R}^3

In this section, we provide one more example of a dispersive blow up, by constructing a solution to the NLS in three dimensions that blows up on the unit sphere. We follow the general strategy and first consider the linear problem. The idea for constructing a linear DBU on a sphere is very simple. Under the radially symmetric assumption, one can easily transform the 3-dimensional linear Schrödinger equation into the 1-dimensional equation. Then, by Proposition 2.1, we construct a DBU at a point on \mathbb{R} , say $r = 1$. Translating it back to the 3-dimensional equation, we obtain a DBU on the unit sphere for the linear problem. Finally, the nonlinear part is controlled via the smoothing estimate Proposition 2.4.

4.1. Construction of a linear DBU. As a first step, we construct a linear DBU that blows up along the unit sphere in \mathbb{R}^3 . Here is a precise statement of the claim.

Lemma 4.1 (Linear DBU on the unit sphere \mathcal{S}^2 in \mathbb{R}^3). *Fix an arbitrary $t_* \in \mathbb{R}$. There exists initial data $u_0 \in C^\infty(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ such that the corresponding global solution $u(t, x)$ to the Cauchy problem (2.1) blows up along the sphere \mathcal{S}^2 , i.e., for any $x_* \in \mathcal{S}^2$, $|u(t, x)| \rightarrow +\infty$ as $(t, x) \rightarrow (t_*, x_*)$. Moreover, $u(t, x)$ is continuous in $\mathbb{R} \setminus \{t_*\} \times \mathbb{R}^3$, and $u(t_*, x)$ is continuous in $\mathbb{R}^3 \setminus \mathcal{S}^2$.*

Proof. We will use radial symmetry to construct such DBU. Suppose that u is a radially symmetric solution to the linear Schrödinger equation:

$$i\partial_t u(t, x) + \Delta u(t, x) = i\partial_t u(t, r) + \partial_r^2 u(t, r) + \frac{2}{r}\partial_r u(t, r) = 0.$$

Let $v(t, r)$ be the even extension of $ru(t, r)$. Then, $v(t)$ solves the 1-dimensional linear Schrödinger equation

$$i\partial_t v + \partial_r^2 v = r \left(i\partial_t u + \partial_r^2 u + \frac{2}{r}\partial_r u \right) = 0. \quad (4.1)$$

Let $v(t, r)$ be the solution to (4.1) with the initial data

$$v_0(r) = \frac{\alpha e^{-\frac{i}{4t_*}|r-1|^2}}{(1+r^2)^m} + \frac{\alpha e^{-\frac{i}{4t_*}|r+1|^2}}{(1+r^2)^m}, \quad m \in \left(\frac{1}{4}, \frac{1}{2} \right]. \quad (4.2)$$

By Proposition 2.1, this initial data in one dimension forms a dispersive blow up at $(t_*, 1)$ (and leads to a solution with the desired continuity properties). Thus, we conclude that for all $|x| = 1$,

$$u(t_*, x) = v(t_*, 1) = \infty.$$

□

4.2. Construction of a nonlinear DBU. We now show that radial initial data

$$u_0(x) = \frac{v_0(|x|)}{|x|}, \quad (4.3)$$

where v_0 is given by (4.2) leads to the solution to the NLS with a DBU on the unit sphere. The proof again relies on the smoothing estimate Proposition 2.4 for controlling the nonlinear term, but first we need to make sure parameters s, p and m are chosen so that $u_0 \in H^s(\mathbb{R}^3)$ and that the dispersive blow up develops.

The following Lemma, which is proven in the Appendix, is useful in finding a sufficient condition for $u_0 \in H^s(\mathbb{R}^3)$.

Lemma 4.2. *Let $u_0(x) = \frac{v_0(|x|)}{|x|}$, where v_0 is given (4.2). If $v_0 \in H^s(\mathbb{R})$ with $s \in [0, \frac{1}{2})$, then $u_0 \in H^s(\mathbb{R}^3)$.*

Now, using Lemma 2.3 to make sure that $v_0 \in H^s(\mathbb{R})$, which in turn ensures that $u_0 \in H^s(\mathbb{R}^3)$, and combining that with Proposition 2.2 and Proposition 2.4 we get

$$\max \left\{ 1 - \frac{1}{p}, \frac{1}{4} \right\} < m \leq \frac{1}{2}. \quad (4.4)$$

For such m to exist, p needs to satisfy $p < 2$. Hence the condition on the order of nonlinearity in the statement of the Theorem 1.5. We are now ready to prove Theorem 1.5.

Proof of Theorem 1.5. Suppose $1 < p < 2$ and let m be such that (4.4) holds. For such an m define initial data u_0 as in (4.3) and let $u(t, x)$ to be the corresponding solution to the NLS. Again, by choosing sufficiently small α (see (4.2)), one may assume that t_* is contained in the interval of existence of u . Then, as a consequence of Lemma 4.1 and Proposition 2.4, the linear part in the integral equation (2.5) blows up at (t_*, x) for all $|x| = 1$, and the nonlinear part remains bounded during the existence time. Therefore, we conclude that $u(t, x)$ blows up at the unit sphere at time t_* . □

Remark 4.3. We restrict our attention to three dimensions, because if $u(t, x)$ is a radial solution to the linear Schrödinger equation in \mathbb{R}^d , then $v(t, r) = r^{\frac{d-1}{2}} u(t, r)$ solves the 1-dimensional Schrödinger equation with an extra term unless $d = 3$.

APPENDIX A. PROOF OF TECHNICAL LEMMA

In the appendix, we prove a technical Lemma 4.2, that provides sufficient condition for the initial data used in the construction of a dispersive blow up on a sphere to be in H^s .

A.1. Initial data for DBU on a sphere. Here we prove Lemma 4.2 which provides a sufficient condition for the radial initial data $u_0(x) = \frac{v_0(|x|)}{|x|}$, where v_0 is given (4.2), to be in $H^s(\mathbb{R}^3)$.

Proof of Lemma 4.2. By changing to spherical coordinates, it is easy to check that

$$\|u_0\|_{L^2(\mathbb{R}^3)} = \left\| \frac{v_0(|x|)}{|x|} \right\|_{L^2(\mathbb{R}^3)} \sim \|v_0\|_{L^2(\mathbb{R})}.$$

Moreover, by the fractional Leibniz rule [8, Proposition 1],

$$\begin{aligned} \|u_0\|_{\dot{H}^s(\mathbb{R}^3)} &= \left\| |\nabla|^s \left(\frac{v_0(|x|)}{|x|} \right) \right\|_{L^2(\mathbb{R}^3)} \leq \left\| \frac{(|\nabla|^s v_0)(|x|)}{|x|} \right\|_{L^2(\mathbb{R}^3)} + \left\| \left(|\nabla|^s \frac{1}{|x|} \right) v_0(|x|) \right\|_{L^2(\mathbb{R}^3)} \\ &\lesssim \| |\nabla|^s v_0 \|_{L^2(\mathbb{R})} + \left\| \frac{v_0(|x|)}{|x|^{1+s}} \right\|_{L^2(\mathbb{R}^3)} \lesssim \| |\nabla|^s v_0 \|_{L^2(\mathbb{R})}, \end{aligned}$$

where in the last step, we used Hölder inequality and Sobolev inequality in Lorentz norms (see [7], for example) to get

$$\left\| \frac{v_0(x)}{|x|^{1+s}} \right\|_{L^2(\mathbb{R}^3)} \sim \left\| \frac{v_0(r)}{|r|^s} \right\|_{L^2(\mathbb{R})} \lesssim \|v_0\|_{L^{\frac{2}{1-2s}, 2}(\mathbb{R})} \lesssim \| |\nabla|^s v_0 \|_{L^2(\mathbb{R})}.$$

□

REFERENCES

- [1] T.B. Benjamin, J.L. Bona and J.J. Mahoney, Model equations for long waves in nonlinear dispersive systems. Philos. Trans. Roy. Soc. London Ser. A 272 (1972), no. 1220, pp. 47-78.
- [2] J.L. Bona, G. Ponce, J.-C. Saut and C. Sparber, Dispersive blow-up for nonlinear Schrödinger equations revisited. J. Math. Pures Appl. (9) 102 (2014), no. 4, 782-811.
- [3] J.L. Bona and J.-C. Saut, Dispersive blowup of solutions of generalized Korteweg-de Vries equations. J. Differential Equations 103 (1993), no. 1, pp. 3-57.
- [4] J.L. Bona and J.-C. Saut, Dispersive blow up II. Schrödinger-type equations, optical and oceanic rogue waves. Chin. Ann. Math. Ser. B 31 (2010), no. 6, pp. 793-818.
- [5] T. Cazenave, Semilinear Schrödinger equations, Courant Lecture Notes in Mathematics. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003.
- [6] D. Foschi, Inhomogeneous Strichartz estimates. J. Hyperbolic Differ. Equ. 2 (2005), no. 1, 1-24.
- [7] L. Grafakos, Classical Fourier analysis. Second edition. Graduate Texts in Mathematics, 249. Springer, New York, 2008. xvi+489 pp.
- [8] J. Nahas and G. Ponce, G, On the persistent properties of solutions to semi-linear Schrödinger equation. Comm. Partial Differential Equations 34 (2009), no. 10-12, 1208-1227.
- [9] H. Pecher, Solutions of semilinear Schrödinger equations in H^s . Ann. Inst. H. Poincaré Phys. Théor. 67 (1997), no. 3, 259-296.
- [10] E.M. Stein, Singular integrals and differentiability properties of functions. Princeton Mathematical Series, No. 30 Princeton University Press, Princeton, N.J. 1970 xiv+290 pp.

Acknowledgements. The work of M.T. was supported in part by the NSF grants DMS-1101192 and DMS-1413064.

THE UNIVERSITY OF TEXAS AT AUSTIN

E-mail address: yhong@math.utexas.edu

E-mail address: mtaskovic@math.utexas.edu